Implicit Differentiation

CHARLES STURT UNIVERSITY

Learning Skills

Introduction:

Expressions of the form $x + y^5 + x^2 + y = 0$ *are differentiated by a method known as implicit differentiation.*

This sheet will teach you to:

- Differentiate by the implicit method

1. How to perform an implicit differentiation

Each term is differentiated with respect to x. When a y term is differentiated, follow the normal rules but because we are differentiating with respect to x a $\frac{dy}{dx}$ must be added as shown below.

Example

Differentiate $x + y^5 + x^2 + y = 0$

Look at each term separately. The instruction to differentiate is shown as $\frac{d}{dx}$

$$\frac{d}{dx}(x) = 1$$
$$\frac{d}{dx}(y^5) = 5y^4\frac{dy}{dx}$$
$$\frac{d}{dx}(x^2) = 2x$$
$$\frac{d}{dx}(y) = 1\frac{dy}{dx} = \frac{dy}{dx}$$
$$\frac{d}{dx}(0) = 0$$

We have now differentiated but need to tidy up and isolate the $\frac{dy}{dx}$ terms

Putting the terms together again:

$$1+5y^4\underline{\frac{dy}{dx}}+2x+\underline{\frac{dy}{dx}}=0$$

remove the 1 and $2x$ from the lhs by subtracting

$$1^{-1}+5y^4\frac{dy}{dx}+2x^{-2x}+\frac{dy}{dx}=0-1-2x$$

$$5y^4\frac{dy}{dx}+\frac{dy}{dx}=-1-2x$$

now factorise the lhs with dy/dx being the common factor

$$\frac{dy}{dx}(5y^4+1)=-1-2x$$

last step divide through by the term in brackets

$$\frac{dy}{dx}=\frac{-1-2x}{(5y^4+1)}$$

Example

Differentiate $x^3y^3+x^2y+2x=12$

There are 4 terms that need to be differentiated. The $2x$ and 12 are straight forward

$$1)\frac{d}{dx}(2x)=2$$

$$2)\frac{d}{dx}(12)=0$$

the other 2 require the product rule

$$3)\frac{d}{dx}\underline{(x^3y^3)}$$

let $u = x^3$ and $v = y^3$

then $u'=3x^2$ and $v' = 3y^2\frac{dy}{dx}$ (remember to add the dy/dx when ever you differentiate a y term)

applying the rule $vu'+uv'$

$$\frac{d}{dx}=y^3\times 3x^2+x^3\times 3y^2\frac{dy}{dx}$$

tidy up

$$\frac{d}{dx}=3x^2y^3+3x^3y^2\frac{dy}{dx}$$

$$4)\ \frac{d}{dx}(x^2y)$$

let $u = x^2$ and $v = y$

then $u' = 2x$ $v' = 1\frac{dy}{dx}$

$$\frac{d}{dx} = y \times 2x + x^2\frac{dy}{dx}$$

$$\text{tidyup}: \quad \frac{d}{dx} = 2xy + x^2\frac{dy}{dx}$$

so now we have differentiated and need to recombine and sort

$$\left[3x^2y^3 + 3x^3y^2\frac{dy}{dx}\right] + \left[2xy + x^2\frac{dy}{dx}\right] + 2 = 0$$

The square brackets are just indicating the differentiation of each of the 4 original terms.

Now group the 2 terms with dy/dx and all other terms change side by subtracting

$$3x^3y^2\frac{dy}{dx} + x^2\frac{dy}{dx} + 3x^2y^3 + 2xy + 2 = 0$$

$$\frac{dy}{dx}(3x^3y^2 + x^2) = -3x^2y^3 - 2xy - 2$$

$$\frac{dy}{dx} = \frac{-3x^2y^3 - 2xy - 2}{3x^3y^2 + x^2}$$

In the sorting process it might be a good idea to use 2 different coloured highlighters, 1 for the terms that have dy/dx attached and the other colour for the other terms.

Example

Find the slope of the tangent at the point (1, 1) to the curve $xy^2 + y^3 = 2$

$$1)\ \frac{d}{dx}(xy^2): u = x, v = y^2, u' = 1, v' = 2y\frac{dy}{dx}$$

$$\text{apply the product rule:} \frac{d}{dx}(xy^2) = y^2 \times 1 + x \times 2y\frac{dy}{dx}$$

$$\text{tidy up:} \frac{d}{dx}(xy^2) = y^2 + 2xy\frac{dy}{dx}$$

$$2)\ \frac{d}{dx}(y^3) = 3y^2\frac{dy}{dx}$$

$$3)\ \frac{d}{dx}(2) = 0$$

Recombining and sorting:

$$y^2 + 2xy\frac{dy}{dx} + 3y^2\frac{dy}{dx} = 0$$

$$\frac{dy}{dx}(2xy + 3y^2) = 0 - y^2$$

$$\frac{dy}{dx} = \frac{-y^2}{2xy + 3y^2}$$

At (1, 1): $\frac{dy}{dx} = \frac{-(1)^2}{2\times1\times1 + 3\times(1)^2} = \frac{-1}{2+3} = \frac{-1}{5} = -\frac{1}{5}$

2. For more information

Visit our Learning Skills website at http://www.csu.edu.au/division/studserv/maths/index.htm

Other useful websites are available at:

http://www.youtube.com/watch?v=23nBGx1avXI